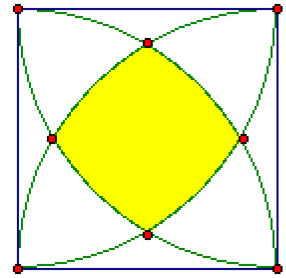
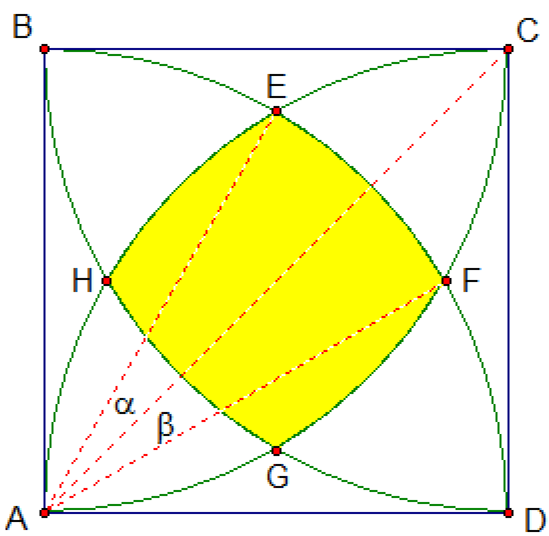
Final Project Part II



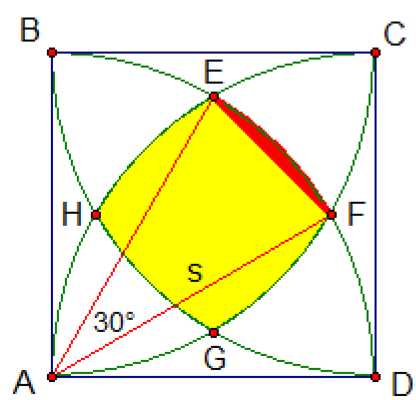
Our final task is to calculate the area of the following region—a square with a quarter circle arc from each corner producing a region bordered by four smaller arcs. I've produced the figure using GSP, the link for which you may find on the website.

My claim is that the area of the yellow region is $\left(1-\sqrt{3}+\frac{\pi}{3}\right)s^2$ if s is the length of the side of the original square.



First, we give ourselves labels to work with and draw in suggestive line segments to break down our figure:

By construction, we see that angle DAC is 45° and angle DAE is 60°, so angle α must be 15°. Likewise, angle BAC is 45° and angle BAF is 60°, so angle β must be 15°. So the central angle for arc EF in the circle centered at A with radius s is 30°, or $\frac{\pi}{6}$ radians.



I can use this information to find the area of the segment of the circle centered at A with radius s determined by arc EF and chord EF as follows:

$$Area_{sector\ EAF}=\frac{1}{2}s^2\left(\frac{\pi}{6}\right)$$

$$Area_{triangle\ AEF}=\frac{1}{2}s^2\sin\left(\frac{\pi}{6}\right)=\frac{s^2}{4}$$

$$Area_{segment\ EF}=Area_{sector\ EAF}-Area_{triangle\ AEF}=\frac{1}{2}s^2\left(\frac{\pi}{6}\right)-\frac{s^2}{4}=\left(\frac{\pi}{12}-\frac{1}{4}\right)s^2$$

Notice, by symmetry, the area of segments FG, HG and HE are the same as EF.

So to find the area of the yellow region, we need only find the area of the square EFGH**. We proceed by finding the length of EF via the Law of Sines: $\frac{\sin 30°}{EF}=\frac{\sin 75°}{s}$. (We know angle FEA is 75° since triangle EAF is isosceles with non-base angle 30°, leaving the equal base angles to sum to 150°.) Solving for EF and evaluating $\sin 75°$ using the sum-formula for sine, we get:

$$EF = \frac{s}{2(\sin 45° \cos 30° + \cos 45° \sin 30°)} = \frac{s}{2\left(\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\right)} = \left(\frac{\sqrt{2}}{\sqrt{3}+1}\right) s$$

So the area of square EFGH is $EF^2 = \left(\frac{2}{4+2\sqrt{3}}\right) s^2 = (2-\sqrt{3})s^2$.

Hence we conclude the area of the yellow region is

$$Area_{square\ EFGH} + 4Area_{segment\ EF} = (2-\sqrt{3})s^2 + \left(\frac{\pi}{3} - 1\right) s^2 = \left(1 - \sqrt{3} + \frac{\pi}{3}\right) s^2.$$

**EFGH is a square by the following observations: Since we know α = β = 15°, we also know angle FAD = 30° and BAE = 30°, so that arc BE = arc EF = arc FD, since they are all on the circle centered at A with radius *s* and are subtended by central angles of equal measure. So angle FED = ½ the measure of arc FD = 15°. A parallel argument for circle centered at D with radius *s* gives angle HEA FD = 15°. Since we know angle AED = 60° by construction, we have angle HEF = angle HEA + angle AEF + angle DEF = 15° + 60° + 15° = 90°. By the symmetry of the picture, we have each angle at E, F, G and H in quadrilateral EFGH is 90° with all side lengths equal to EF, i.e. EFGH is a square